COMPLEX MULTIPLICATION

1. The modular function j

In this section we study modular functions on $\mathfrak{h}$, these correspond to meromorphic functions on the modular curve $Y(1)$ but with a special condition at ∞.

1.1. **Fourier coefficients.** Consider the function

$$z \mapsto e^{2\pi i z}$$

It is a surjection from $\mathfrak{h}$ to the punctured disc

$$D_0 = \{z \in \mathbb{C} : 0 < |z| < 1\}$$

moreover it is locally biholomorphic (bijective and holomorphic with holomorphic inverse). We define a new variable q by setting

$$q = e^{2\pi i z}$$

Suppose now f is a meromorphic function on $\mathfrak{h}$ such that

$$f(z) = f(z+n), \forall n \in \mathbb{Z}$$

then the value of f only depends on $q = e^{2\pi i z}$ so that there exists a unique meromorphic function

$$F : D_0 \to \mathbb{C}$$

such that

$$f(z) = F(q)$$

We can consider 0 as a singularity of F, and if this singularity is at worst a pole, then F has a Laurent series expansion about 0:

$$F(q) = \sum_{n>>\infty} a_n q^n$$

Definition 1.1. Let f be a meromorphic function on $\mathbb{C}$ such that

$$f(z) = f(z+n), \quad \forall n \in \mathbb{Z}$$

Then f is holomorphic (resp. meromorphic) at ∞ if the function $F(q)$ defined above is holomorphic (resp. meromorphic) at 0.

If f is meromorphic at ∞ the *Fourier series expansion* of f or its *q-expansion* is the power series $F(q) = \sum_{n>>0} a_n q^n$.

Recall the group $\Gamma = SL_2(\mathbb{Z})$ acts on $\mathfrak{h}$. Suppose f is a meromorphic function on $\mathfrak{h}$ such that $f(\gamma z) = f(z)\ \forall \gamma \in \Gamma$, then in particular since $\zeta z = z + 1$ so that $f(z) = (z+n), \quad \forall n \in \mathbb{Z}$ so we can apply the above constructions

Definition 1.2. A modular function is a meromorphic function $f : \mathfrak{h} \to \mathbb{C}$ satisfying the following two properties:

i) $f(\gamma z) = f(z)\ \forall \gamma \in \Gamma$

ii) f is holomorphic at ∞.

The space of all modular functions will be denoted $\mathcal{M}_0'$.

It is straightforward to check that $\mathcal{M}'_0$ is a field and contains all the constant functions.

Remark 1.3. If we add a point ∞ to the Riemann surface $Y(1) = \Gamma\backslash\mathfrak{h}$, where we consider ∞ as the center of the disc $\mathbb{D}_0$ in the new coordinate q. The new coordinate q defines a chart around this point, hence we may consider $Y(1)\coprod\{\infty\}$ as Riemann surface which we denote by $X(1)$. This construction is analogous to the adding ∞ to $\mathbb{C}$ to obtain $\mathbb{P}^1(\mathbb{C})$ and one sees that $X(1)$ is also compact.

1.2. **Modular functions and modular forms.** The main theorem of this section gives a very simple description of $\mathcal{M}_0$.

Theorem 1.4. *There exists a unique modular function j satisfying the following conditions:*

i) j is holomorphic on $\mathfrak{h}$, has a simple pole at ∞ and $j(\omega) = 0$

ii) j induces by passage to the quotient a bijection $\Gamma\backslash\mathfrak{h} \to \mathbb{C}$

iii) The first term in the q-expansion of j is $\frac{1}{q}$

Moreover we have an isomorphism $\mathcal{M}'_0 = \mathbb{C}(j)$, i.e. any modular function is a rational function of j.

In order to construct j we need to consider a larger class of functions than modular functions.

Definition 1.5. Let $k \in \mathbb{N}$. A modular function of weight k is a meromorphic function on $\mathfrak{h}$ satisfying the following two properties

i) $f(\gamma z) = (cz + d)^{2k} f(z)$

ii) f is meromorphic at ∞

If in addition f is holomorphic and holomorphic at ∞ then we say f is a *modular form*. If f is 0 at ∞, i.e. the q-expansion has all non-zero coefficients in positive degree, then f is a *a cusp form*. We will use $\mathcal{M}'_k, \mathcal{M}_k$ and $\mathcal{S}_k$ to denote the spaces of modular functions, modular forms and cusp forms of weight k respectively.

Note that condition i) in the above definition ensures $f(z) = f(z + n), \quad \forall n \in \mathbb{Z}$ so that condition *ii*) makes sense. Then modular functions of weight 0 are just modular functions in the sense of Definition 1.2.

Exercise: Let f and g be modular functions of weights k_1 and k_2 respectively. Show that fg is a modular function of weight k_1+k_2 and f/g is a modular function of weight $k_1 - k_2$ ($g \neq 0$) and that if f and g are both modular forms, then fg is also a modular form.

The exercise shows that if we let

$$\mathcal{M} := \bigoplus_{i=0}^{\infty} \mathcal{M}_k$$

then $\mathcal{M}$ has a natural structure of a graded ring.

Remark 1.6. Strictly speaking these should be modular functions of weight k and level Γ. However since we do not consider other levels in this course, we will ignore this point.

Recall the lattice $\Lambda_\tau = \langle 1, \tau \rangle$ of the last lecture. For $k \geq 2$ an integer, define the *Eisenstein series of weight* $2k$ to be the function

$$E_k(z) := \sum_{\omega \in \Lambda_z - 0} \frac{1}{\omega^{2k}} = \sum_{(m,n) \in \mathbb{Z}^2 - (0,0)} \frac{1}{(nz+m)^{2k}}$$

We also define the following functions

$$g_2(z) = 60E_2(z), \quad g_2(z) = 140E_3(z), \quad \Delta = g_2^3(z) - 27g_3^2(z)$$

Proposition 1.7. *For $k \geq 2$ an integer*

i) The series defining E_k above converges to a holomorphic function on $\mathfrak{h}$. Moreover $E_k(z)$ is a modular form of weight $2k$.

ii) $E_k(\infty) = 2\zeta(2k)$ where ζ is the Riemann zeta function.

iii) The function

$$j := 1728 \frac{g_2^3}{\Delta}$$

is a modular function (of weight 0)

Proof. i) As each of the terms in the series is a holomorphic function on $\mathfrak{h}$ it suffices to prove that the series converges absolutely and uniformly on compact subset of $\mathfrak{h}$. Let P_n denote the parallelogram with vertices $\{\pm nz, \pm n\}$ so that $\Lambda \cap P_n$ has $8n$ points. Let r be the minimum distance from 0 to a point on the parallelogram P_1, we then have the estimate

$$\sum_{\omega \in \Lambda_z - 0} \frac{1}{|\omega|^{2k}} = \sum_{n=1}^{\infty} \sum_{\omega \in \Lambda_z \cap P_n} \frac{1}{|\omega|^{2k}} < \sum_{n=1}^{\infty} \frac{8n}{(nr)^{2k}}$$
$$= \frac{8}{r^{2k}} \sum_{n=1}^{\infty} \frac{1}{n^{2k-1}} = \frac{8}{r^{2k}} \zeta(2k-1)$$

For z in a compact subset of $\mathfrak{h}$, the value r has a positve infimum, and since $\zeta(2k-1)$ coverges for $k > 1$, it follows easily that the series converges absolutely and uniformly in any compact subset of $\mathfrak{h}$.

We leave it as an exercise to check that $E_k(z)$ has the required translation property with respect to elements of Γ.

For holomorphicty at ∞, this is equivalent to checking that $E_k(z)$ is bounded as $z \to \infty$. But this follows from the above estimate by noting that for $C > 0$, the value r has an infimum on the set $\{z \in \mathfrak{h} : |z| > C\}$.

ii) By absolute convergence we may rearrange the order of summation so that

$$E_k(z) = \sum_{n \in \mathbb{Z} - 0} \frac{1}{n^{2k}} + \sum_{m \in \mathbb{Z} - 0} \sum_{n \in \mathbb{Z}} \frac{1}{mz+n}$$

All the terms in the right term tend to 0 as $\Im z \to \infty$, and the first term is just $2\zeta(2k)$.

iii) Since g_2 has weight 2 and g_3 has weight 3, then Δ is a modular form of weight 6 which is non-zero by Corollary 1.9. It follows from the exercise that j is a modular function of weight 0.

□

Remarkably, we will actually show that the above construction in some sense exhausts all modular forms (of any weight). More precisely, we will show that the space of modular forms $\mathcal{M}$ is isomorphic to $\mathbb{C}[E_2, E_3]$. This is something of a miracle as what the statement is really saying is that there are many algebraic relations between these modular forms, whose definitions a priori are based completely in analysis.

The following is the key result we need to prove the above statements. Given a meromorphic function f and $z \in \mathbb{C}$ we denote by $v_z(f)$ the order of the function f at z. If f is a modular function, then $v_\infty(f)$ is defined in the obvious way, and it follows from the isolation of zeros/poles theorem that there are only finitely many $z \in D \cup \{\infty\}$ for which $v_z(f)$ is non-zero.

Proposition 1.8. *Let f be a modular function of weight k and $e_z = |\Gamma_z|/2$, then we have the relation*

$$v_\infty(f) + \sum_{z \in \Gamma \backslash \mathfrak{h}} \frac{v_z(f)}{e_z} = \frac{k}{6}$$

Proof. Let C be the path in the diagram on page 87 of Serre's "A Course in Arithmetic." It folllows from the argument principle that

$$\frac{1}{2\pi i} \int \frac{f'}{f} dz = \sum_{z \in \text{interior of } C} v_z(f)$$

Now we compute the left hand side of the expressions. The integrals along the segments AB and $D'E$ cancel since $f(z) = f(z+1)$. To compute the contribution of EA, we note that changing the variable to q, this segment is mapped to a path which loops once around 0, hence taking $z \to \infty$ this contribution will be $-v_\infty(f)$.

For the contribution DD' at $\omega = \frac{1+\sqrt{-3}}{2}$, note that as we take the arc smaller and smaller, we may assume that the only possible zero/pole occurs at ω. Also for a small enough arc, the value of the f'/f on the arc is roughly equal on the arc, so that in the limit

$$\frac{1}{2\pi i} \int_{DD'} \frac{f'}{f} dz \to -\frac{1}{6} v_\omega(f)$$

The $-$ sign comes about from the orientation of the arcs, the same remarks reply for the point i and ω^2. The contribution along BB' is equal to the one along DD' and the contribution along CC' is $-\frac{v_i(f)}{2}$.

Finally we must compute the contribution along the arcs $B'C$ and $C'D$. Let

$$\mu = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

then μ maps $B'C$ to $C'D$ with the opposite orientation. The relation

$$f(\mu z) = f(-1/z) = z^{2k} f(z)$$

gives us

$$\frac{f'(-1/z)}{z^2} = z^{2k} f'(z) + 2kz^{2k-1} f(z)$$

We thus obtain

$$\int_{DC'} \frac{f'}{f} dz = -\int_{BC'} \frac{f'(\mu z)}{f(\mu z)} d\mu z$$

$$= -\int_{BC'} \frac{f'(z)}{f(z)} + \frac{2k}{z} dz$$

Now $\int_{BC'} \frac{2k}{z} dz$ can be computed explicitly, by eg. making the substitution $z = e^{2\pi i x}$, and we obtain $\frac{k\pi i}{3}$ for this integral. Taking the limit for smaller and smaller arcs the formula above becomes:

$$\frac{k}{6} - v_\infty(f) - \frac{v_i(f)}{2} - \frac{v_\omega(f)}{3} = \sum_{z \in \text{interior } D} v_z(f)$$

The result follows from the the fact that $e_\omega = 3$ and $e_i = 2$ and $e_z = 1$ for all z in the interior of D. □

Corollary 1.9. *i) E_2 and E_3 can only have zeros at the points ω or i. Δ is non-zero on $\mathfrak{h}$ and has zero of order 1 at ∞.*
ii) $\mathcal{M} = \oplus_{k=0}^{\infty} \mathcal{M}_k \cong \mathbb{C}[E_2, E_3]$ as rings.

Proof. i) Since E_2 and E_3 are holomorphic, it follows that they have non-negative orders everywhere, hence since $k/6 < 1$, it follows they can only have zeros at the specified points.

By part ii) of Proposition 1.7 we have

$$g_2(\infty) = 60E_2(\infty) = 120\zeta(4) = 4/3\pi^4$$

$$g_3(\infty) = 140E_3(\infty) = 280\zeta(6) = 8/27\pi^6$$

Thus $v_\infty(\Delta) \geq 1$ and since $\Delta \in \mathcal{M}_6$, the above formula shows $v_\infty(\Delta) = 1$ and that it cannot have any other zeros.

ii) The map

$$\mathcal{M}_k \to \mathbb{C}$$

given by $f \mapsto f(\infty)$ has kernel $\mathcal{S}_k$ and is surjective for $k \geq 6$. The surjectivity follows from the fact that we can write $k = 2\alpha + 3\beta$ for some positive integers α, β. Then $E_2^\alpha E_3^\beta \in \mathcal{M}_k$, is non-zero at ∞. Therefore $\mathcal{M}_k = \mathcal{S}_k \oplus \langle E_2^\alpha E_3^\beta \rangle$, so that it suffices to show $\mathcal{S}_k$ is spanned by E_2 and E_3.

Let $f \in \mathcal{S}_k$, then $f/\Delta \in \mathcal{M}_k$ since Δ is non-zero on $\mathfrak{h}$ and has a simple pole at 0. Thus by induction we need to show for $k = 0, 1, 2, 3, 4, 5$ that $\mathcal{M}_k \subset \mathbb{C}[E_2, E_3]$.

We leave this easy exercise, which follows from Proposition 1.8 for the reader to check.

□

Exercise: Use Proposition 1.8 to show that $\mathcal{M}_k \subset \mathbb{C}[E_2, E_3]$ for $k = 0, ..., 5$

Proof. (of Theorem 1.4) It is clear any function is unique as the difference would be a holomorphic map on a compact Riemann surface, hence constant and since $j(\omega) = 0$ this constant must be 0. Thus it suffices to show j satisfies the conditions of the theorem.

i) This follows immediately from Corollary 1.9.

ii) We need to show for $a \in \mathbb{C}$, $j(z) - a$ has a zero of order 1. Since $j(z) - a$ is a modular function of weight 0 with a simple pole at ∞, Proposition 1.8 shows that

$$\sum_{z \in \Gamma \backslash \mathfrak{h}} \frac{v_z(j-a)}{e_z} = 1$$

It follows that $j - a$ has a unique zero on $\Gamma \backslash \mathfrak{h}$.

iii) This follows from an explicit computation, see Serre's "A course in Arithmetic."

We now show that any $f \in \mathcal{M}_0'$ is a rational function of j. Suppose $f \in \mathcal{M}_0'$ has poles at a. We may multiply f by a suitable power of $j - j(a)$ so that f is holomorphic at a, hence we may assume f is holomorphic on $\Gamma\backslash\mathfrak{h}$. Suppose f has a pole of order n at ∞, then $f\Delta^n \in \mathcal{M}_{6n}$, but we know $\mathcal{M}_{6n}$ is spanned by $E_2^\alpha E_3^\beta$ where $2\alpha + 3\beta = 6n$. Thus it suffices to show the result for $\frac{E_2^\alpha E_3^\beta}{\Delta^n}$.

But $2\alpha + 3\beta = 6n$ implies $3|\alpha$ and $2|\beta$, so that it suffices to show the result for $\frac{E_2^3}{\Delta}$ and $\frac{E_3^2}{\Delta}$. This follows from the identities

$$\frac{E_2^3}{\Delta} = \frac{g_2^3}{60^3\Delta} = \frac{1}{1728.60^3}j$$

$$\frac{E_3^2}{\Delta} = \frac{g_3^2}{140^2.\Delta} = \frac{1}{140^2.27}\left(\frac{g_2^3}{\Delta} - 1\right) = \frac{1}{1728.140.27}j - \frac{1}{140^2.27}$$

□